ECE 330 Exam #2, Fall 2018 Name: Solution
90 Minutes

Section (Check One) MWF 9am _______ MWF 10am _______

1. ________ / 25 2. ________ / 25

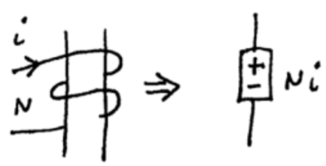
3. ________ / 25 4. ________ / 25 Total ________ / 100

Useful information

$\sin(x) = \cos(x - 90^\circ)$ $\quad \overline{V} = \overline{Z}\overline{I}$ $\quad \overline{S} = \overline{V}\overline{I}^*$ $\quad \mu_0 = 4\pi \cdot 10^{-7}$ H/m

$\int_C \mathrm{H} \cdot dl = \int_S \mathrm{J} \cdot n\,da$ $\quad \int_C \mathrm{E} \cdot dl = -\frac{\partial}{\partial t}\int_S \mathrm{B} \cdot n\,da$ $\quad MMF = Ni = \phi\Re$

$\Re = \frac{l}{\mu A}$ $\quad B = \mu H$ $\quad \phi = BA$ $\quad \lambda = N\phi$ $\quad \lambda = Li$ (if linear)




$W_m = \int_0^{\lambda} i\,d\hat{\lambda}$ $\quad W_m' = \int_0^{i} \lambda\,d\hat{i}$ $\quad W_m + W_m' = \lambda i$ $\quad f^e = \frac{\partial W_m'}{\partial x} = -\frac{\partial W_m}{\partial x}$ $\quad x \to \theta$

$f^e \to T^e$

$\underset{a\to b}{EFE} = \int_a^b i\,d\lambda$ $\quad \underset{a\to b}{EFM} = -\int_a^b f^e\,dx$ $\quad \underset{a\to b}{EFE} + \underset{a\to b}{EFM} = W_{mb} - W_{ma}$ $\quad \lambda = \frac{\partial W_m'}{\partial i}$ $\quad i = \frac{\partial W_m}{\partial \lambda}$

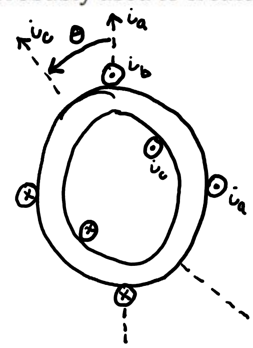
$$\dot{x}_1 = f_1(x_1, x_2) \text{ and } \dot{x}_2 = f_2(x_1, x_2)$$

$$x(t_0 + \Delta t) \approx x(t_0) + \Delta t \cdot \left.\frac{dx}{dt}\right|_{t=t_0}$$

## Problem 1A (7 points)

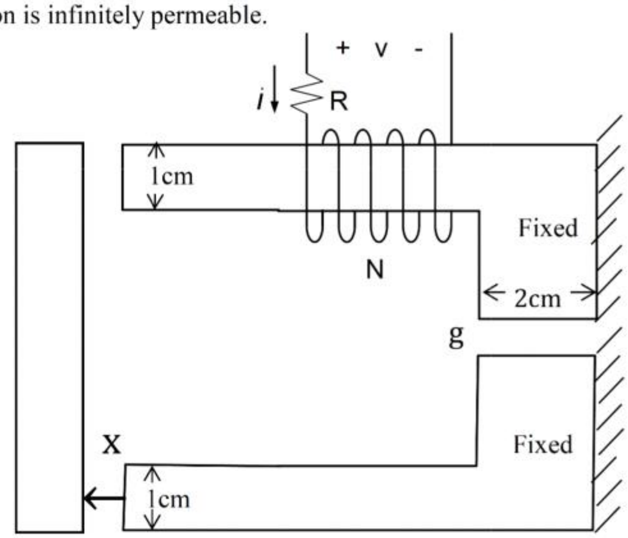
For the system shown below, draw the physical system that was probably used to create this model (show the coil orientation and the angle definition):

$$\lambda_a = L_a i_a + M\cos\theta\, i_c$$
$$\lambda_b = L_b i_b + M\sin\theta\, i_c$$
$$\lambda_c = L_c i_c + M\cos\theta\, i_a + M\sin\theta\, i_b$$

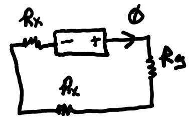



## Problem 1B (18 points)

For the structure drawn below, the movable member is constrained to move left and right only as indicated in the figure where "$x$" is the distance to the edge of the movable member. The large members are fixed, and the depth into the page for all members is 2cm. The gap $g$ is 1mm, and the number of turns N =100. You may neglect fringing in all the gaps, and you may assume the iron is infinitely permeable.




a)




$R_g = \frac{g}{\mu_0 A_g}$ $\quad R_x = \frac{x}{\mu_0 A_x}$

$A_g = 2A_x$

$R_g = \frac{g}{2\mu_0 A_x}$

$R_{tot} = R_g + 2R_x \Rightarrow R_{tot} = \frac{g}{2\mu_0 A_x} + \frac{2x}{\mu_0 A_x}$

$= \frac{1}{2\mu_0 A_x}(g+4x)$

$R_{tot} = \frac{g}{2\mu_0 A_x}\left(1+4\left(\frac{x}{g}\right)\right)$

$\boxed{R_{tot} = 1.989\times10^{6}(1+4000x)}$

a) Find the total reluctance of the main flux path in terms of x
b) Find the flux linkage, λ (defined for the voltage polarity shown)
c) Find an expression for the voltage, v in terms of x, i, and time t

b) $\phi = \frac{Ni}{R_{tot}}$

$\lambda = N\phi$

$\lambda = \frac{N^2 i}{R_{tot}}$

$\boxed{\lambda = \frac{5.03\times10^{-3}}{1+4000x}\, i}$

c) $v = iR + \frac{d\lambda}{dt}$

$\frac{d\lambda}{dt} = \frac{5.03\times10^{-3}}{1+4000x}\frac{di}{dt} - \frac{5.03\times10^{-3}}{(1+4000x)^2}(4000)\frac{dx}{dt}\, i$

$\boxed{v = i\left(R - \frac{20.12}{(1+4000x)}\frac{dx}{dt}\right) + \frac{5.03\times10^{-3}}{1+4000x}\frac{di}{dt}}$

(Extra page at the end if you need it)

Problem 2 (25 Points)

A stator-rotor system has flux linkage given as

$$\lambda_s = L_s\left(1+\cos\left(2\theta\right)\right)i_s + M\left(9\cos\left(\theta\right)+\cos\left(3\theta\right)\right)i_r$$
$$\lambda_r = M\left(9\cos\left(\theta\right)+\cos\left(3\theta\right)\right)i_s + L_r\left(1+\cos\left(2\theta\right)\right)i_r$$

a) Find the co-energy $W_m'$.

b) Find the torque of electric origin $T^e$.

c) What is the torque when $L_s = L_r = 1$H, $M = 1.3$H, $i_s = 1$A, $i_r = 0.1$A, and $\theta = -45°$?

a) $W_m' = \int_0^{i_s} \lambda_s(\hat{i}_s, i_r=0, \theta)\,d\hat{i}_s + \int_0^{i_r} \lambda_r(i_s, \hat{i}_r, \theta)\,d\hat{i}_r$

$= \int_0^{i_s} L_s(1+\cos(2\theta))\hat{i}_s\,d\hat{i}_s + \int_0^{i_r}\left[M(9\cos(\theta)+\cos(3\theta))i_s + L_r(1+\cos(2\theta))\hat{i}_r\right]d\hat{i}_r$

$W_m' = \frac{1}{2}L_s(1+\cos(2\theta))i_s^2 + M(9\cos(\theta)+\cos(3\theta))i_s i_r + \frac{1}{2}L_r(1+\cos(2\theta))i_r^2$

$\boxed{W_m' = \frac{1}{2}\left[L_s i_s^2 + L_r i_r^2\right](1+\cos(2\theta)) + M(9\cos(\theta)+\cos(3\theta))i_s i_r}$

b) $T^e = \frac{\partial W_m'}{\partial \theta}$

$T^e = \frac{1}{2}\left[L_s i_s^2 + L_r i_r^2\right](-2\sin(2\theta)) + M(-9\sin(\theta) - 3\sin(3\theta))i_s i_r$

$\boxed{T^e = -\left[L_s i_s^2 + L_r i_r^2\right]\sin(2\theta) - M(9\sin(\theta)+3\sin(3\theta))i_s i_r}$

c) $T^e = -\left[1(1)^2 + 1(0.1)^2\right]\sin(2(-45°)) - 1.3(9\sin(-45°)+3\sin(3(-45°)))(1)(0.1)$

$T^e = 1.01 + 1.103$

$\boxed{T^e = 2.113 \text{ Nm}}$

(Extra page at the end if you need it)

Problem 3 (25 Points)

A certain system has flux linkage given as

$$\lambda = \frac{0.3}{x-0.01} i^2 \qquad \text{with the constraint that } x > 0.01.$$

Find the energy from the mechanical system (EFM) and the energy from the electrical system (EFE) as the system moves from $x = 0.020$ m to $x = 0.015$ m while $i$ is held constant at $i = 2$ A.

$$\underset{a\to b}{EFM} = \int_a^b -f^e dx$$

$$f^e = \frac{\partial W_m'}{\partial x} \Rightarrow f^e = \frac{-0.1}{(x-0.01)^2} i^3$$

$$W_m' = \int_0^i \lambda \, d\hat{i} \Rightarrow W_m' = \int_0^i \frac{0.3}{x-0.01} \hat{i}^2 d\hat{i} \Rightarrow W_m' = \frac{0.1}{x-0.01} i^3$$

$$\underset{a\to b}{EFM} = \int_{0.02}^{0.015} \frac{0.1}{(x-0.01)^2} (2)^3 dx \Rightarrow \underset{a\to b}{EFM} = \int_{0.02}^{0.015} \frac{0.8}{(x-0.01)^2} dx \Rightarrow \underset{a\to b}{EFM} = \frac{-0.8}{x-0.01}\Bigg|_{0.02}^{0.015}$$

$$\underset{a\to b}{EFM} = -0.8\left[\frac{1}{0.015-0.01} - \frac{1}{0.02-0.01}\right]$$

$$\underset{a\to b}{EFM} = -0.8[200-100]$$

$$\boxed{\underset{a\to b}{EFM} = -80 \text{ J}}$$

$$W_m + W_m' = \lambda i$$

$$W_m = \lambda i - W_m'$$

$$W_m = \frac{0.3}{x-0.01} i^3 - \frac{0.1}{x-0.01} i^3$$

$$W_m = \frac{0.2}{x-0.01} i^3$$

$$W_{ma} = \frac{0.2}{0.02-0.01}(2)^3 \Rightarrow W_{ma} = 160 \text{ J}$$

$$W_{mb} = \frac{0.2}{0.015-0.01}(2)^3 \Rightarrow W_{mb} = 320 \text{ J}$$

$$\underset{a\to b}{EFE} + \underset{a\to b}{EFM} = W_{mb} - W_{ma}$$

$$\underset{a\to b}{EFE} = 320 - 160 - (-80)$$

$$\boxed{\underset{a\to b}{EFE} = 240 \text{ J}}$$

(Extra page at the end if you need it)

Problem 4 (25 Points)

A spring pendulum has equations of motion given as

$$m\ddot{r} = -k\left(r - R_0\right) + mg\cos\left(\theta\right) + mr\dot{\theta}^2$$

$$mr\ddot{\theta} = -mg\sin\left(\theta\right) - 2m\dot{r}\dot{\theta}$$

where $m$ is the mass of the attached object, $R_0$ is the unstretched length of the spring, and the dot notation signifies a time derivative, i.e. $\frac{dr}{dt} = \dot{r}$.

For this given system,

a) Find the equilibrium positions.
b) Rewrite the equations of motion in state space form
c) If the spring pendulum initially starts at $r(0) = R_0$, $\dot{r}(0) = 0$, $\theta(0) = \frac{\pi}{12}$, $\dot{\theta}(0) = 0$, use $\Delta t = 0.001$ s to determine the state variables at $t = 0.002$ s using $R_0$=1 m, $k$=200 N/m, $m$=2 kg, and $g$=9.81 m/s$^2$.

a) at equilibrium: $\dot{r} = \ddot{r} = 0$
$\dot{\theta} = \ddot{\theta} = 0$

$0 = -k(r - R_0) + mg\cos(\theta)$

$0 = -mg\sin(\theta)$

$\sin(\theta) = 0$

$\theta = 0, \pi$

$\underline{\theta = 0:}$

$0 = -k(r - R_0) + mg$

$k(r - R_0) = mg$

$r - R_0 = \frac{mg}{k}$

$r = R_0 + \frac{mg}{k}$

$\underline{\theta = \pi:}$

$0 = -k(r - R_0) - mg$

$k(r - R_0) = -mg$

$r - R_0 = -\frac{mg}{k}$

$r = R_0 - \frac{mg}{k}$

$$\boxed{\begin{array}{l}\theta = 0,\ r = R_0 + \frac{mg}{k} \\ \theta = \pi,\ r = R_0 - \frac{mg}{k}\end{array}}$$

b) $\ddot{r} = -\frac{k}{m}(r - R_0) + g\cos(\theta) + r\dot{\theta}^2$

$\ddot{\theta} = -\frac{g}{r}\sin(\theta) - 2\frac{\dot{r}\dot{\theta}}{r}$

$\frac{dr}{dt} = \dot{r}$

$\frac{d\theta}{dt} = \dot{\theta}$

$\frac{d\dot{r}}{dt} = -\frac{k}{m}(r - R_0) + g\cos(\theta) + r\dot{\theta}^2$

$\frac{d\dot{\theta}}{dt} = -\frac{g}{r}\sin(\theta) - 2\frac{\dot{r}\dot{\theta}}{r}$

$x_1 = r,\ x_2 = \theta$
$x_3 = \dot{r},\ x_4 = \dot{\theta}$

$\frac{dx_1}{dt} = x_3$

$\frac{dx_2}{dt} = x_4$

$\frac{dx_3}{dt} = -\frac{k}{m}(x_1 - R_0) + g\cos(x_2) + x_1 x_4^2$

$\frac{dx_4}{dt} = -\frac{g}{x_1}\sin(x_2) - 2\frac{x_3 x_4}{x_1}$

c)

| $t$ | $r$ | $\theta$ | $\dot{r}$ | $\dot{\theta}$ |
|---|---|---|---|---|
| 0 | 1 | 0.262 | 0 | 0 |
| 0.001 | 1 | 0.262 | 0.009475 | -0.002541 |
| 0.002 | 1.000009475 | 0.262 | 0.01895 | -0.005082 |